ON HARMONIOUS CHROMATIC NUMBER OF TRIPLE STAR GRAPH

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Abstract. A Harmonious coloring of a graph $G$ is a proper vertex coloring of $G$, in which every pair of colors appears on at most one pair of adjacent vertices and the harmonious chromatic number of graph $G$ is the minimum number of colors needed for the harmonious coloring of $G$ and it is denoted by $X_H(G)$. The purpose of this paper is to extend the double star graph [12] and to discuss harmonious coloring for central graph, middle graph and total graph of extended double star graph i.e. triple star graph.

Key Words: Central graph, Middle graph, Total graph, Harmonious coloring and Harmonious chromatic number.

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1. Introduction

Let $G$ be a finite undirected graph with vertex set $V(G)$ and edge set $E(G)$ having no loops and multiple edges.

All graphs considered here are undirected. In the whole paper, the term coloring will be used to define vertex coloring of graphs. A proper coloring of a graph $G$ is the coloring of the vertices of $G$ such that no two neighbors in $G$ are assigned the same color.

A Harmonious coloring of a graph $G$ is a proper vertex coloring of $G$, in which every pair of colors appears on at most one pair of adjacent vertices and the harmonious chromatic number of graph $G$ is the minimum number of colors needed for the harmonious coloring of $G$ and it
is denoted by $X_H(G)$. The purpose of this paper is to introduce the triple star graph and to discuss the harmonious coloring of triple star graph families.

The first paper on harmonious graph coloring was published in 1982 by Frank Harary and M. J. Plantholt [6]. However, the proper definition of this notion is due to J.E. Hopcroft and M. S. Krishnamoorthy [5] in 1983. A collection of articles in harmonious coloring can be found in the bibliography [3].

In 2012 M. Venkatachalam, J. Vernold Vivin and K. Kaliraj [12] discussed Harmonious Coloring on double star Graph Families. In this paper we extended the double star graph [12] which is known as triple star graph and discuss harmonious coloring for this graph families.

2. Definitions

Definition 2.1. The central graph \([2,3,7,9,11,12,13]\) of a graph is obtained by subdividing each edge of \(G\) exactly once and joining all the non adjacent vertices of \(G\).

Definition 2.2. The middle graph \([2,3,4,9,10,11,12,13]\) of \(G\), denoted by \(M(G)\), has the vertex set of \(M(G)\) is \(V(G)\cup E(G)\). Two vertices \(x, y\) in the vertex set of \(M(G)\) are adjacent in \(M(G)\) in case one of the following holds:

(a) \(x, y\) are in \(E(G)\) and \(x, y\) are adjacent in \(G\).
(b) \(x\) is in \(V(G)\), \(y\) is in \(E(G)\), and \(x, y\) are incident in \(G\).

Definition 2.3. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). The total graph \([2,3,7,11,12,13]\) of \(G\) is denoted by \(T(G)\) and is defined as follows.

The vertex set of \(T(G)\) is \(V(G)\cup E(G)\). Two vertices \(x, y\) in the vertex set of \(T(G)\) is adjacent in \(T(G)\), if one of the following holds:

(a) \(x, y\) are in \(V(G)\) and \(x\) is adjacent to \(y\) in \(G\).
(b) \(x, y\) are in \(E(G)\) and \(x, y\) are adjacent in \(G\).
(c) \(x\) is in \(V(G)\), \(y\) is in \(E(G)\) and \(x, y\) are incident in \(G\).

Definition 2.4. Triple star \(K_{1,n,n,n}\) is a tree obtained from the double star \([12]\) \(K_{1,n,n}\) by adding a new pendant edge of the existing \(n\) pendant vertices. It has \(3n + 1\) vertices and \(3n\) edges.

Let \(V(K_{1,n,n,n}) = \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, \ldots, w_n\} \cup \{u_1, u_2, \ldots, u_n\}\) and \(\text{E}(K_{1,n,n,n}) = \{e_1, e_2, \ldots, e_n\} \cup \{e_1', e_2', \ldots, e_n'\} \cup \{e_1'', e_2'', \ldots, e_n''\}\).
3. Harmonious Coloring of Triple Star Graph Families

**Theorem 3.1.** For any triple star graph $K_{1,n,n,n}$ the harmonious chromatic number, $X_H(C(K_{1,n,n,n})) = 4n + 3$.

**Proof** First we apply the definition of central graph on $K_{1,n,n,n}$. Let the edge $v_i w_i$ and $w_i u_i (1 \leq i \leq n)$ of $K_{1,n,n,n}$ be subdivided by the vertices $e_i (1 \leq i \leq n)$, $e'_i (1 \leq i \leq n)$ and $e''_i (1 \leq i \leq n)$.

It is clear that

\[
V(C(K_{1,n,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\} \\
\cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{e''_i : 1 \leq i \leq n\}.
\]

The vertices $v_i, (1 \leq i \leq n)$ induce a clique (largest complete subgraph) of order $n$ (say $K_n$) and the vertices $v, u_i (1 \leq i \leq n)$ induce a clique (largest complete subgraph) of order $n+1$ (say $K_{n+1}$) in $C(K_{1,n,n,n})$ respectively (see figure 2). Also we observe that the number of edges in $C(K_{1,n,n,n})$ is $(9n^2 + 9n)/2$.

Thus we have $X_H(C(K_{1,n,n,n})) \geq 4n + 3$.

Now we apply the colors to the vertices of $C(K_{1,n,n,n})$ as follows: Taking color class $C = \{c_1, c_2, c_3, ..., c_{4n+3}\}$.

(i) For $(1 \leq i \leq n)$, assign the color $c_i$ to $u_i$. 

Fig 1. Triple Star Graph
(ii) For $(1 \leq i \leq n)$, assign the color $c_{n+i}$ to $w_i$.
(iii) For $(1 \leq i \leq n)$, assign the color $c_{2n+i}$ to $v_i$.
(iv) For $(1 \leq i \leq n)$, assign the color $c_{3n+i}$ to $e_i$.
(v) For $(1 \leq i \leq n)$, assign the color $c_{4n+1}$ to $e_i'$ and color $c_{4n+2}$ to $e_i''$ and at last assign the color $c_{4n+3}$ to $v$.

Therefore $X_H(C(K_{1,n,n,n})) \leq 4n + 3$. Hence $X_H(C(K_{1,n,n,n})) = 4n + 3$.

![Fig 2. C(K1,n,n,n) with coloring](image)

**Theorem 3.2.** For any triple star graph $K_{1,n,n,n}$ the harmonious chromatic number, $X_H(M(K_{1,n,n,n})) = 3n + 3$ for $n > 1$.

**Proof**  First we apply the definition of middle graph. Let the edge $vv_i$, $v_iw_i$ and $w_iu_i$ $(1 \leq i \leq n)$ of $K_{1,n,n,n}$ be subdivided by the vertices $e_i$ $(1 \leq i \leq n)$, $e_i'(1 \leq i \leq n)$ and $e_i''(1 \leq i \leq n)$.

It is clear that

$$V(M(K_{1,n,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e_i' : 1 \leq i \leq n\} \cup \{e_i'' : 1 \leq i \leq n\}.$$  

The vertices $v, e_i(1 \leq i \leq n)$ induce a clique of order $n+1$ (say $K_{n+1}$) in $M(K_{1,n,n,n})$ (see figure 3). Also we observe that the number of edges in $M(K_{1,n,n,n})$ is $(n^2 + 15n)/2$.

Thus we have $X_H(M(K_{1,n,n,n})) \geq 3n + 3$. 


Now we apply the colors to the vertices of $M(K_{1,n,n,n})$ as follows:

Taking color class $C = \{c_1, c_2, c_3, ..., c_{3n+3}\}$.

(i) For $(1 \leq i \leq n)$, assign the color $c_1$ to $u_i$ and $v$.
(ii) For $(1 \leq i \leq n)$, assign the color $c_{1+i}$ to $e_i$.
(iii) For $(1 \leq i \leq n)$, assign the color $c_{n+1+i}$ to $e_i'$.
(iv) For $(1 \leq i \leq n)$, assign the color $c_{2n+1+i}$ to $e''_i$.
(v) At last for $(1 \leq i \leq n)$, assign the color $c_{3n+2}$ to $v_i$, color $c_{3n+3}$ to $w_i$.

Therefore $X_H(M(K_{1,n,n,n})) \leq 3n + 3$. Hence $X_H(M(K_{1,n,n,n})) = 3n + 3$.

Theorem 3.3. For any triple star graph $K_{1,n,n,n}$ the harmonious chromatic number $X_H(T(K_{1,n,n,n})) = 4n + 2$.

Proof. First we apply the definition of total graph. Let the edge $vv_i$, $v_i w_i$ and $w_i u_i$ $(1 \leq i \leq n)$ of $K_{1,n,n,n}$ be subdivided by the vertices $e_i$ $(1 \leq i \leq n)$, $e'_i (1 \leq i \leq n)$ and $e''_i (1 \leq i \leq n)$.

It is clear that

\[
V(T(K_{1,n,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\} \\
\cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{e''_i : 1 \leq i \leq n\}.
\]

The vertices $v, e_i (1 \leq i \leq n)$ induce a clique of order $n + 1$ (say $K_{n+1}$) in $T(K_{1,n,n,n})$ (see figure 4). Also we observe that the number of edges in $T(K_{1,n,n,n})$ is $(n^2 + 21n)/2$.

Thus we have $X_H(T(K_{1,n,n,n})) \geq 4n + 2$. 

Fig 3. $M(K_{1,n,n,n})$ with coloring
Now we apply the colors to the vertices of $T(K_{1,n,n,n})$ as follows: Taking color class $C = \{c_1, c_2, c_3, \ldots, c_{4n+2}\}$.

(i) For $((1 \leq i \leq n)$, assign the color $c_i$ to $e_i$.
(ii) For $(1 \leq i \leq n)$, assign the color $c_{n+i}$ to $v_i$.
(iii) For $(1 \leq i \leq n)$, assign the color $c_{2n+i}$ to $w_i$.
(iv) For $(1 \leq i \leq n)$, assign the color $c_{3n+1+i}$ to $e’_i$.
(v) For $(1 \leq i \leq n)$, assign the color $c_{3n+1+i}$ to $e’’_i$.
(vi) At last for $(1 \leq i \leq n)$, assign the color $c_{4n+2}$ to $u_i$, $v$.

Therefore $X_H(T(K_{1,n,n,n})) \leq 4n + 2$. Hence $X_H(M(K_{1,n,n,n})) = 4n + 2$.

Fig 4. $T(K_{1,n,n,n})$ with coloring

4. Conclusion

In this paper, we introduce triple star graph and discuss the harmonious coloring and find the harmonious chromatic number for central graph, middle graph and total graph of triple star graph $4n + 3$, $3n + 3$ and $4n + 2$ respectively.

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