**Abstract.** Intuitionistic Logic was introduced by L. E. J. Brouwer in [1] and Heyting algebra was defined by A. Heyting to formalize the Brouwer’s intuitionistic logic [4]. The concept of Heyting algebra has been accepted as the basis for intuitionistic propositional logic. Heyting algebras have had applications in different areas. The co-Heyting algebra is the same lattice with dual operation of Heyting algebra [5]. Also, co-Heyting algebras have several applications in different areas.

In this paper, we introduced the new concept $HH^*-\text{Intuitionistic Heyting Valued }\Omega\text{-Algebra}$. The purpose of introducing this new concept is to expand the field of researchers’ area using both membership degree and non-membership degree. This allows us to get more sensitive results. The $HH^*-\text{Intuitionistic Heyting valued set}$, $HH^*-\text{Intuitionistic Heyting valued relation}$, $HH^*-\text{Intuitionistic Heyting valued }\Omega\text{-algebra}$ and the homomorphism over $HH^*-\text{Intuitionistic Heyting valued }\Omega\text{-algebra}$ were defined.

**Key Words:** Heyting Valued Algebra, co-Heyting Valued Algebra, Omega Algebra, Intuitionistic Logic.

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1. Introduction

A Heyting algebra is a lattice expanded with a implication operation $\rightarrow$. In 1930, the concept of Heyting algebra introduced by A. Heyting [4] as following,
Definition 1.1. [4] A Heyting algebra is an algebra \((H, \vee, \wedge, \rightarrow, 0_H, 1_H)\) such that \((H, \vee, \wedge, 0, 1)\) is a lattice and for all \(a, b, c \in H\),

\[ a \leq b \rightarrow c \iff a \wedge b \leq c \]

\((H, \vee, \wedge, 0_H, 1_H)\) is a Heyting algebra with \(\forall a, b \in H\),

\[ a \rightarrow b = \bigvee \{ c : a \wedge c \leq b, c \in H \} \].

The notion of co-Heyting algebra for a Heyting algebra defined in [5].

Definition 1.2. [5] A co-Heyting algebra is an algebra \((H^*, \vee, \wedge, \hookrightarrow, 0_{H^*}, 1_{H^*})\) such that \((H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})\) is a lattice and for all \(a, b, c \in H^*\),

\[ a \hookrightarrow b \leq c \iff a \leq b \vee c \]

\((H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})\) is a co-Heyting algebra with \(\forall a, b \in H^*\),

\[ a \hookrightarrow b = \bigwedge \{ c : a \vee c \geq b, c \in H^* \} \].

The implication operation in co-Heyting algebra is define \(a \rightarrow b = b \hookrightarrow a\).

Let \((H, \vee, \wedge, \rightarrow, 0_H, 1_H)\) be a Heyting algebra and \((H^*, \vee, \wedge, \hookrightarrow, 0_{H^*}, 1_{H^*})\) is a co-Heyting algebra of \((H, \vee, \wedge, \rightarrow, 0_H, 1_H)\).

\(L = H \times H^*\) is a lattice with \((x_1, x_2) \leq_L (y_1, y_2) \iff x_1 \leq_H y_1\) and \(x_2 \geq_{H^*} y_2\). For \(x, y \in L, x = (x_1, x_2)\) and \(y = (y_1, y_2)\), the operators \(\wedge\) and \(\vee\) on \((L, \leq_L)\) are defined as following;

\[ x \wedge y = (\min \{x_1, y_1\}, \max \{x_2, y_2\}) \]
\[ x \vee y = (\max \{x_1, y_1\}, \min \{x_2, y_2\}) \]

\(0_L = (0_H, 1_{H^*})\) and \(1_L = (1_H, 0_{H^*})\) are called greatest and least element of \(L\), respectively.

The \(\rightarrow_L\) is a binary operation on \(L\) with \(x \rightarrow_L y = (x_1 \rightarrow_H y_1, y_2 \hookrightarrow_{H^*} x_2)\).

Definition 1.3. A \(HH^*\)-Intuitionistic Heyting algebra is an algebra \((L, \vee, \wedge, \rightarrow_L, 0_L, 1_L)\) such that \((L, \vee, \wedge, 0_L, 1_L)\) is a lattice as defined above and for all \(a, b, c \in L\),

\[ a \leq_L b \rightarrow_L c \iff a \wedge b \leq_L c. \]
The \( (L, \lor, \land, 0_L, 1_L) \) lattice is a \( HH^* \)-Intuitionistic Heyting algebra with \( \forall a, b \in L, a = (a_1, a_2) \) and \( b = (b_1, b_2) \),

\[
a \rightarrow_L b = \lor \{ c : a \land c \leq_L b, c \in L \}
\]

\[
= \left( \lor \{ c_1 : a_1 \land c_1 \leq_H b_1, c_1 \in H \}, \land \{ c_2 : b_2 \lor c_2 \geq a_2, c_2 \in H^* \} \right)
\]

**Proposition 1.4.** An algebra \( (L, \lor, \land, \rightarrow_L, 0_L, 1_L) \) is a \( HH^* \)-Intuitionistic Heyting algebra if and only if \( (L, \lor, \land, 0_L, 1_L) \) is an lattice and the following identities hold for all \( a, b, c \in L \),

\( 1 \) \( a \rightarrow_L a = 1_L \)
\( 2 \) \( a \land (a \rightarrow_L b) = a \land b \)
\( 3 \) \( b \land (a \rightarrow_L b) = b \)
\( 4 \) \( a \rightarrow_L (b \land c) = (a \rightarrow_L b) \land (a \rightarrow_L c) \)

**Proof.** \( 1 \) \( \forall a \in L, \)

\[
a \rightarrow_L a = (a_1 \rightarrow_H a_1, a_2 \leftrightarrow_H a_2) = (1_H, 0_{H^*}) = 1_L
\]

\( 2 \) From definition it is obtained that,

\[
a \land (a \rightarrow_L b) = (a_1 \land (a_1 \rightarrow_H b_1), a_2 \lor (b_2 \leftrightarrow_H a_2))
\]

\[
= (a_1 \land b_1, a_2 \lor b_2) = a \land b
\]

\( 3 \) \( \forall a, b \in L, \)

\[
b \land (a \rightarrow_L b) = (b_1 \land (a_1 \rightarrow_H b_1), b_2 \lor (b_2 \leftrightarrow_H a_2)) = b
\]

\( 4 \) \( \forall a, b, c \in L, \)

\[
a \rightarrow_L (b \land c) = (a_1 \rightarrow_H (b_1 \land c_1), (b_2 \lor c_2) \leftrightarrow_H a_2)
\]

\[
= ((a_1 \rightarrow_H b_1) \land (a_1 \rightarrow_H c_1), (b_2 \leftrightarrow_H a_2) \lor (c_2 \leftrightarrow_H a_2))
\]

\[
= (a \rightarrow_L b) \land (a \rightarrow_L c)
\]

\( \Box \)

2. \( HH^* \)-Intuitionistic Valued Sets

In this section, firstly we introduced the concept of \( HH^* \)-Intuitionistic Valued Set and \( HH^* \)-Intuitionistic Valued Function. Then, we defined \( HH^* \)-Intuitionistic valued equivalence relation and equivalence class. Some properties of these concepts were examined.
Definition 2.1. Let $H$ be a complete Heyting algebra and $H^*$ be a complete co-Heyting algebra of $H$. Let $X$ be a universal and $L = H \times H^*$ then $HH^*$–Intuitionistic valued set is determined with $[=]$ function

$$[=]_L : X \times X \to L, [=]_L (a, b) = ([a = b]_H, [a = b]_{H^*})$$

which satisfy the following conditions.

1. $[a = b]_L \leq_L [b = a]_L$
2. $[a = b]_L \land [b = c]_L \leq_L [a = c]_L$

If $X$ a universal and $[=]_L$ is a function satisfy the above conditions then $X$ called $HH^*$–Intuitionistic valued set and it is shown $(X, [=]_L)$.

Let $X$ be a universal. $u \in X, E(u)$ means the degree of existence the element $u$. For $HH^*$–Intuitionistic valued set we will use,

$$E(u) = [u \in X]_L.$$ 

So, $[u \in X]_L = [u = u]_L$.

Definition 2.2. Let $A$ be a $HH^*$–Intuitionistic valued set. The subset of $A$ is a $s : A \to L$ function with following conditions.

1. $[x \in s]_L \land [x = y]_L \leq_L [y \in s]_L$
2. $[x \in s]_L \leq_L [x \in A]_L$

Definition 2.3. Let $(X, [=]_L)$ and $(Y, [=]_L)$ are $HH^*$–Intuitionistic valued sets. If $f_L : X \times Y \to L$ function satisfy the following conditions then called $HH^*$–Intuitionistic valued function and it is shown $f_L : X \to Y$.

\[ F1 \quad f_L(x, y) \leq_L [x = x]_L \land [y = y]_L \]
\[ F2 \quad [x = x']_L \land f_L(x, y) \land [y = y']_L \leq_L f_L(x', y') \]
\[ F3 \quad f_L(x, y) \land f_L(x, y) \leq_L [y = y']_L \]
\[ F4 \quad [x = x]_L \leq_L \lor \{ f_L(x, y) : y \in Y \} \]

Notation 2.4. $f_L(x, y) := [f_L(x) = y]_L = ([f_L(x) = y]_H \land [f_L(x) = y]_{H^*})$

Definition 2.5. Let $(X, [=]_L)$ be an $HH^*$–Intuitionistic valued set. $I : X \times X \to L$, $I(x, x') = [x = x']_L$ function is called unit function.

Definition 2.6. Let $(X, [=]_L), (Y, [=]_L)$ and $(Z, [=]_L)$ are $HH^*$–Intuitionistic valued sets and $f_L : X \to Y$, $g_L : Y \to Z$ are $HH^*$–Intuitionistic valued functions. For $x \in X, z \in Z$,

$$(g \circ f)_L(x, z) = \lor \{ f_L(x, y) \land g_L(y, z) : y \in Y \}.$$
Proposition 2.7. Let $(X, =_L), (Y, =_L)$ and $(Z, =_L)$ be $HH^*$–Intuitionistic valued sets and $f_L : X \rightarrow Y$, $g_L : Y \rightarrow Z$ are $HH^*$–Intuitionistic valued functions. The function $(g \circ f)_L : X \rightarrow Z$ is a $HH^*$–Intuitionistic valued function.

Proof. (i) Let $x \in X, z \in Z, (g \circ f)_L (x, z) = \bigvee \{ f_L(x, y) \land g_L(y, z) : y \in Y \}$.

$$\bigvee \{ f_H(x, y) \land g_H(y, z) : y \in Y \} \leq \bigvee \{ [x = x]_H \land [y = y]_H \land [z = z]_H : y \in Y \}$$

$$= [x = x]_H \land [z = z]_H \land \bigvee \{ [y = y]_H : y \in Y \}$$

$$\leq [x = x]_H \land [z = z]_H$$

and

$$\bigwedge \{ f_H^*(x, y) \lor g_H^*(y, z) : y \in Y \} \geq \{ [x = x^*]_H \lor [y = y^*]_H \lor [z = z^*]_H : y \in Y \}$$

$$= [x = x^*]_H \lor [z = z^*]_H \lor \{ [y = y^*]_H : y \in Y \}$$

$$\geq [x = x^*]_H \lor [z = z^*]_H$$

So, $(g \circ f)_L (x, z) \leq_L [x = x]_L \land [z = z]_L$

(ii) Let $x, x' \in X$ and $z, z' \in Z$,

$$[x = x']_L \land (g \circ f)_L (x, z) \land [z = z']_L = [x = x']_L \land \bigvee \{ f_L(x, y) \land g_L(y, z) : y \in Y \} \land [z = z']_L$$

Firstly,

$$[x = x']_H \land \bigvee \{ f_H(x, y) \land g_H(y, z) : y \in Y \} \land [z = z']_H$$

$$= \bigvee \{ [x = x']_H \land f_H(x, y) \land g_H(y, z) \land [z = z']_H : y \in Y \}$$

$$= \bigvee \{ [x = x']_H \land f_H(x, y) \land [y = y]_H \land g_H(y, z) \land [z = z']_H : y \in Y \}$$

$$\leq \bigvee \{ f_H(x, y') \land g_H(y', z') : y' \in Y \}$$

on the other hand,

$$[x = x']_H \lor \{ f_H^*(x, y) \lor g_H^*(y, z) : y \in Y \} \lor [z = z']_H$$

$$= \{ [x = x^*]_H \lor f_H^*(x, y) \lor g_H^*(y, z) \lor [z = z^*]_H : y \in Y \}$$

$$= \{ [x = x^*]_H \lor f_H^*(x, y) \lor [y = y^*]_H \lor g_H^*(y, z) \lor [z = z^*]_H : y \in Y \}$$

$$\geq \{ f_H^*(x', y') \lor g_H^*(y', z') : y' \in Y \}$$

Therefore, $[x = x']_L \land (g \circ f)_L (x, z) \land [z = z']_L \leq (g \circ f)_L (x', z')$

(iii) Let $x \in X$ and $z, z' \in Z$,

$$(g \circ f)_L (x, z) \land (g \circ f)_L (x, z')$$

$$= \bigvee \{ f_L(x, y) \land g_L(y, z) : y \in Y \} \lor \bigvee \{ f_L(x, t) \land g_L(t, z') : t \in Y \}$$

Now,

$$\bigvee \{ f_H(x, y) \land g_H(y, z) : y \in Y \} \lor \bigvee \{ f_H(x, t) \land g_H(t, z') : t \in Y \}$$

$$= \bigvee \{ f_H(x, y) \land f_H(x, y) \land g_H(y, z) \land g_H(y, z') : y \in Y \}$$

$$= \bigvee \{ [y = y]_H \land [z = z']_H : y \in Y \}$$

$$\leq [z = z']_H$$
and
\[
\{f_H^*(x, y) \lor g_H^*(y, z) : y \in Y\} \lor \{f_H^*(x, t) \lor g_H^*(t, z') : t \in Y\} \\
= \{f_H^*(x, y) \lor f_H^*(x, y) \lor g_H^*(y, z) \lor g_H^*(y, z') : y \in Y\} \\
\geq \{[y = y]_{H^*} \lor [z = z']_{H^*} : y \in Y\} \\
\geq [z = z']_{H^*}
\]

(iv) Let \(x \in X\), then
\[
\lor \{(g \circ f)_L(x, z) : z \in Z\} = \lor \{(f_L(x, y) \land g_L(y, z) : y \in Y) : z \in Z\}
\]
\[
= \lor \{f_H(x, y) : y \in Y\} \lor \lor \{g_H(y, z) : y \in Y\} : z \in Z\}
\geq [x = x]_H \land \lor \{g_H(y, z) : y \in Y\} : z \in Z\}
= [x = x]_H \land \lor \{g_H(y, z) : z \in Z\} : y \in Y\}
\geq [x = x]_H \land \lor \{y = y\}_H : y \in Y\}
\]
and
\[
\{f_H(x, y) \lor g_H(y, z) : y \in Y\} : z \in Z\}
= \{f_H(x, y) \lor g_H(y, z) : y \in Y\} \lor \{g_H(y, z) : y \in Y\} : z \in Z\}
\leq [x = x]_H \lor \{g_H(y, z) : y \in Y\} : z \in Z\}
= [x = x]_H \lor \{g_H(y, z) : z \in Z\} : y \in Y\}
\leq [x = x]_H \lor \{y = y\}_H : y \in Y\}
\]
So, \(\lor \{(g \circ f)_L(x, z) : z \in Z\} \geq [x = x]_L \quad \square\)

**Definition 2.8.** Let \((X, =_L)\) and \((Y, =_L)\) are \(HH^*-\)Intuitionistic valued sets and \(f_L : X \to Y\) is \(HH^*-\)Intuitionistic valued function.

1. \(f_L\) is a monomorphism. \(\iff \forall x, x' \in X, y \in Y, \) \(f_L(x, y) \land f_L(x', y) \leq [x = x']\)
2. \(f_L\) is an epimorphism. \(\iff \forall y \in Y, \) \([y = y']_L \leq \{f_L(x, y) : x \in X\}\)

**Definition 2.9.** Let \((X, =_L)\) be a \(HH^*-\)Intuitionistic valued sets. \(R_L : X \times X \to L\) is called \(HH^*-\)Intuitionistic valued equivalence relation if and only if

R1 \(R_L(x, y) \land [x = x]_L = R_L(x, y), R_L(x, y) \land [y = y]_L = R_L(x, y)\)
R2 \(R_L(x, y) \land [x = x']_L \leq R_L(x, x'), R_L(x, y) \land [y = y']_L \leq R_L(x, y)\)
Example 2.10. Let \((X, =_L)\), \((Y, =_L)\) are \(HH^*\)–Intuitionistic valued sets and \(f_L : X \rightarrow Y\) is \(HH^*\)–Intuitionistic valued function. \(\forall x_1, x_2 \in X,
\)
\[
C_f(x_1, x_2) = [f_L(x_1) = f_L(x_2)]_L
\]
\[
= ([f_H(x_1) = f_H(x_2)]_H, [f_{H^*}(x_1) = f_{H^*}(x_2)]_{H^*})
\]
function is a \(HH^*\)–Intuitionistic valued equivalence relation on \(X\).

Definition 2.11. Let \(R_L\) be a \(HH^*\)–Intuitionistic valued equivalence relation on \(X\). \(d_L : X \rightarrow L\) is called equivalence class of \(R_L\) ⇔
\[
d_1 \ d_L(x) \land R_L(x, x') \leq d_L(x')
\]
\[
d_2 \ d_L(x) \land d_L(y) \leq R_L(x, y)
\]
d\(d_L(x)\) is the equivalence class of \(x \in X\).

Proposition 2.12. Let \(R_L\) be a \(HH^*\)–Intuitionistic valued equivalence relation on \(X\) and \(\sigma_L, \tau_L\) are equivalence class of \(R_L\).
\[
\bigvee \{\sigma_L(x) : x \in X\} = \bigvee \{\tau_L(x) : x \in X\} \text{ and } \sigma_L(x) \leq \tau_L(x) \Rightarrow \sigma_L = \tau_L
\]

Proof. For \(x_0 \in X, \tau_L(x_0) = \bigvee \{\tau_L(x) \land \sigma_L(x) : x \in X\}.
\[
\bigvee \{\tau_H(x) \land \sigma_H(x) : x \in X\} \leq \bigvee \{R_H(x_0, x) \land \sigma_H(x)\} \leq \sigma_H(x_0)
\]
and
\[
\{\tau_{H^*}(x) \lor \sigma_{H^*}(x) : x \in X\} \geq \{R_H(x_0, x) \lor \sigma_{H^*}(x) : x \in X\} \geq \sigma_{H^*}(x_0)
\]
\[
\square
\]

Proposition 2.13. Let \((X, =_L), (Y, =_L)\) are \(HH^*\)–Intuitionistic valued sets and \(f_L : X \rightarrow Y\) is \(HH^*\)–Intuitionistic valued function. \(f_L\) is surjective ⇔ \(\forall y \in Y, \bigvee \{[f_L(x) = y] : x \in X\} = [y = y]_L\).
3. \(HH^*\)-Intuitionistic Valued \(\Omega\)-Algebras

To create \(HH^*\)-Intuitionistic Valued \(\Omega\)-Algebra, let \(\Omega\) be a set of the operations defined as follows.

Let \(L = H \times H^*\) such that \(H\) is a complete Heyting algebra and \(H^*\) is a complete co-Heyting algebra of \(H\), so

\[\Omega = \{\omega_L : X^n \times X \rightarrow L : \omega \text{ satisfy F1-F4 conditions}\}\]

It means that, if \(\omega \in \Omega\), \(\omega\) is \(HH^*\)-Intuitionistic valued function. The concept of \(HH^*\)-Intuitionistic valued \(\Omega\)-algebra can be defined as following:

**Definition 3.1.** Let \(\langle X, =_{L} \rangle\) \(HH^*\)-Intuitionistic valued set. \(A = \langle X, \Omega \rangle\) is called \(HH^*\)-Intuitionistic valued \(\Omega\)-algebra if the following condition satisfy.

For \(\omega_L \in \Omega\) and \(\{(x_1, x_2, ..., x_n), c\} \in X^n \times X\),
\[
\forall \{\{x_i \in A\}_L \land \omega_L ((x_1, x_2, ..., x_n), d) \lor [c = d]_L : i = 1, 2, ..., n\} : d \in X \}
\[
\geq \omega_L ((x_1, x_2, ..., x_n), c)
\]

**Example 3.2.** Let \(A = \langle X, \Omega \rangle\) be a \(HH^*\)-Intuitionistic valued \(\Omega\)-algebra.

\[\{\Theta\} : A \rightarrow L, [x \in \{\Theta\}]_L = 1_L\]

is a subset of \(A\).
\[E = \langle \{\Theta\} , \Omega \rangle\]

is a \(HH^*\)-Intuitionistic valued \(\Omega\)-algebra. \(E\) is called trivial \(HH^*\)-Intuitionistic valued \(\Omega\)-algebra.

**Definition 3.3.** Let \(A = \langle X, \Omega \rangle\) be a \(HH^*\)-Intuitionistic valued \(\Omega\)-algebra. If \(K \subseteq X, B : K \rightarrow L\) is \(HH^*\)-Intuitionistic valued set, \(B = \langle A, =_{L} \rangle \subseteq \langle A, =_{L} \rangle\) and for all \(\omega_L \in \Omega, \omega_L \downarrow_{B}\) satisfy the (1) then \(B\) is \(HH^*\)-Intuitionistic valued \(\Omega\)-subalgebra of \(A\).

**Example 3.4.** Let \(A = \langle X, \Omega \rangle\) be a \(HH^*\)-Intuitionistic valued \(\Omega\)-algebra. \(E = \langle \{\Theta\} , \Omega \rangle\) is \(HH^*\)-Intuitionistic valued \(\Omega\)-subalgebra.

**Definition 3.5.** Let \(A = \langle X, \Omega \rangle\) and \(B = \langle Y, \Omega \rangle\) be similar \(HH^*\)-Intuitionistic valued \(\Omega\)-algebras and \(f_L : A \rightarrow B\) be \(HH^*\)-Intuitionistic valued function. \(f_L\) is a \(HH^*\)-Intuitionistic valued \(\Omega\)-algebra homomorphism if

1. \(H1\) \([x = x]_L = [f_L(x) = f_L(x)]_L\)
2. \(H2\) \([x = x']_L \leq [f_L(x) = f_L(x')]_L\)
3. \(H3\) \(f_L(\omega_L (x_1, x_2, ..., x_n), y) = \{f_L (x_i, y_i) : y = \omega_L (y_1, y_2, ..., y_n), f_L (x_i, y_i) > 0\}\)
Example 3.6. Let $A = (X, \Omega)$ be a $HH^*$–Intuitionistic valued $\Omega$–algebra and $f_L : E \to A$, $g_L : E \to A$ are $HH^*$–Intuitionistic valued functions. 

$\forall x, I (\{\Theta\}, x) = 1_L$ $HH^*$–Intuitionistic valued $\Omega$– algebra homomorphism exist. This homomorphism is unique.

Proposition 3.7. Let $A, B, C$ are similar $HH^*$–Intuitionistic valued $\Omega$– algebras. If $f_L : A \to B$, $g_L : B \to C$ are $HH^*$–Intuitionistic valued $\Omega$– algebra homomorphisms then $(g \circ f)_L : A \to C$ is a $HH^*$–Intuitionistic valued $\Omega$– algebra homomorphism.

Proof. Let $x_1, x_2, \ldots, x_n \in A, z \in C$,

i. $\quad [x = x]_L = \bigl( [x = x]_H, [x = x]_{H^*} \bigr) = \bigl( [f_H(x) = f_H(x)]_H, [f_{H^*}(x) = f_{H^*}(x)]_{H^*} \bigr) = [f_L(x) = f_L(x)]_L$

ii. $\quad [x = x']_L = \bigl( [x = x']_H, [x = x']_{H^*} \bigr) \leq \bigl( [f_H(x) = f_H(x')]_H, [f_{H^*}(x) = f_{H^*}(x')]_{H^*} \bigr) = [f_L(x) = f_L(x')]_L$

iii. $(g \circ f)_L (\omega_L (x_1, x_2, \ldots, x_n), z) = \bigvee \{ f_L (\omega_L (x_1, x_2, \ldots, x_n), y) \land g_L (y, z) : y \in B \}$

Hence,

$\forall \{ \{ f_H (x_i, y_i) : y = \omega_H (y_1, y_2, \ldots, y_n), f_H (x_i, y_i) > 0 \} \land g_H (y, z) : y \in B \} = \bigvee \left\{ \begin{array}{l} \{ f_H (x_i, y_i) : y = \omega_H (y_1, y_2, \ldots, y_n) \land f_H (x_i, y_i) > 0 \} : y \in B \end{array} \right\}$

$= \bigvee \left\{ \begin{array}{l} \{ f_H (x_i, y_i) > 0 \} \land g_H (y, z) : y \in B \end{array} \right\}$

and

$= \bigvee \left\{ \begin{array}{l} \{ f_{H^*} (x_i, y_i) : y = \omega_{H^*} (y_1, y_2, \ldots, y_n), f_{H^*} (x_i, y_i) > 0 \} \land g_{H^*} (y, z) \end{array} \right\}$

Now, we obtain that

$(g \circ f)_L (\omega_L (x_1, x_2, \ldots, x_n), z) = \bigvee \{ (g \circ f)_L (x_i, z_i) : z = \omega_L (z_1, z_2, \ldots, z_n), (g \circ f)_L (x_i, z_i) > 0 \}$. \[\square\]
**Definition 3.8.** Let $A = \langle X, \Omega \rangle$ be a $HH^*$–Intuitionistic valued $\Omega$–algebra and $R_L$ be a $HH^*$–Intuitionistic valued equivalence relation on $X$. $R_L$ called $HH^*$–Intuitionistic valued congruence relation if and only if $\forall \omega_L \in \Omega, x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in X^n,$

$$R_L(\omega_L(x), \omega_L(y)) = \{R_L(x_i, y_i) : i = 1, \ldots, n\}$$

**Example 3.9.** Let $(X, =_L), (Y, =_L)$ are $HH^*$–Intuitionistic valued $\Omega$–algebras and $f_L : X \rightarrow Y$ is $HH^*$–Intuitionistic valued $\Omega$– algebra homomorphism. $\forall x_1, x_2 \in X, C_f(x_1, x_2)$ is a $HH^*$–Intuitionistic valued congruence relation on $X$.

Let $(X, =_L)$ be an $HH^*$–Intuitionistic valued set and $X/R_L$ is set of the equivalence classes of $R_L, HH^*$–Intuitionistic valued equivalence relation on $X$. For, $\tau_L, \sigma_L \in X/R_L, [\tau_L = \sigma_L]_L$ defined as following,

$$[\tau_L = \sigma_L]_L = \bigvee \{\tau_L(x) \lor \sigma_L(x) : x \in X\}$$

So, $(X/R_L, =_L)$ is $HH^*$–Intuitionistic valued sets.

If $d_1, d_2, \ldots, d_n, \tau \in X/R_L$ then $HH^*$–Intuitionistic valued operator on $X/R_L$ defined as follow:

$$\omega_L((d_1, d_2, \ldots, d_n), \tau) = [d_{\omega_L(x_1,x_2,\ldots,x_n)} = \tau]_L$$

Therefore, $a \in X$,

$$d_{\omega_L(x_1,x_2,\ldots,x_n)}(a) = \{d_{x_i}(a) : i = 1, \ldots, n\}$$

**Theorem 3.10.** Let $A = \langle X, \Omega \rangle$ be a $HH^*$–Intuitionistic valued $\Omega$–algebra, $R_L$ be a $HH^*$–Intuitionistic valued equivalence relation on $X$ and $\langle X/R_L, \Omega \rangle$ be a $HH^*$–Intuitionistic valued $\Omega$–algebra. The function,

$$f_L(a) = \varphi_a : X \times X/R_L \rightarrow L, \varphi_a(x, d_a) = d_a(x) \text{ for } a \in X$$

is a $HH^*$–Intuitionistic valued epimorphism from $X$ to $X/R_L$ and $X/R_L$ is uniquely determined.

**Proof.** It is clear that, $\varphi_a$ is surjective and $X/R_L$ is uniquely determined.

Now, let $a, x \in X$.

$$\varphi_a(x, d_a) = d_a(x) = (\tau_a(x), \sigma_a(x)) \in L,$$

$$\tau_a(x) = \bigvee \{R_H(x, y) \land [x = a]_H : y \in X\}$$

$$\geq \bigvee \{R_H(x, a) \land R_H(a, y) \land [x = a]_H : y \in X\}$$

$$\geq \bigvee \{R_H(a, y) \land [x = a]_H : y \in X\} = \tau_x(a)$$
and
\[ \sigma_a(x) = \{RH^*(x,y) \lor [x = a]_{H^*} : y \in X\} \leq \{RH^*(x,a) \lor RH^*(a,y) \lor [x = a]_{H^*} : y \in X\} \leq \{RH^*(a,y) \lor [x = a]_{H^*} : y \in X\} = \sigma_x(a) \]

So, \( d_a(x) = d_x(a) \). Furthermore,

\[ \varphi_{\omega_L(x_1,x_2,\ldots,x_n)}(x, d_{\omega_L(x_1,x_2,\ldots,x_n)}(x)) = \bigvee \{d_{x_i}(x) : i = 1,\ldots,n\} = \bigvee \{\varphi_{x_i}(x, d_{x_i}) : i = 1,\ldots,n\} \]

\( \varphi_a \) is homomorphism.

\[ \square \]

4. Conclusion

Thanks to this extension, we can study algebraic properties of lattice valued sets in broad perspective. We can examine the kind of \( HH^* \)-Intuitionistic valued \( \Omega \)-algebra homomorphisms, can be defined generated \( HH^* \)-Intuitionistic valued \( \Omega \)-subalgebra, filters in \( HH^* \)-Intuitionistic Valued \( \Omega \)-Algebra. Furthermore, the concept free \( HH^* \)-Intuitionistic Valued \( \Omega \)-Algebra can be studied.

References


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