DERIVED METABELIAN GROUPS FROM $H_v$-GROUPS

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Abstract. In this paper first we introduce and analyze a new definition of left and right commutators in $H_v$-group. Secondly, using commutators we introduce a new strongly equivalence relation $\pi^*$ on an $H_v$-group $H$ such that the quotient $H/\pi^*$, the set of all equivalence classes, is a metabelian group. Then we introduce metabelian $H_v$-groups and investigate some of their properties. Finally, we investigate some properties of commutators for the class of weak polygroups.

Key Words: $H_v$-group, metabelian group, metabelian $H_v$-group, weak polygroup.

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1. Introduction

In [12] Vougioklis introduced the notion of $H_v$—groups as a generalization of the notion of hypergroups. In general, the motivation for $H_v$—group is the following: we know that the quotient of a group with respect to a normal subgroup is a group. In 1934 F. Marty states that, the quotient of a group with respect to any subgroup is a hypergroup (see [9]). Vougioklis states that the quotient of a group with respect to any partition is an $H_v$—group. Since then the study of $H_v$—structures has been continued in many directions by T. Vougiouklis, B. Davvaz, S. Spartalis, A. Dramalidis, S. Hoskova, and some other mathematicians. We invite the readers for more study about hyperstructures theory and
derived metabelian groups from $H_v$-groups 59

its applications to [2, 3, 4, 5, 11]. The first fundamental relation defined on hypergroups is the $\beta^*$-relation, introduced by Koskas [8] in 1970, in connection with the heart of a hypergroup and studied mainly by Corsini, Davvaz, Freni, Leoreanu, Vougiouklis. Later on, Freni [7] introduced the $\gamma$-relation on a hypergroup, as a generalization of the relation $\beta$, proving that $\gamma^*$ is the smallest regular relation on a semihypergroup such that the corresponding quotient is a commutative semigroup. In the class of hyperrings, several fundamental relations have been defined till now with respect to both (hyper)operations (addition and multiplication) for example see [6, 10, 14]. H. Aghabozorgi et.al., introduced and analyzed the notions of left and right commutators in polygroups and they provided a detailed structure description of derived subpolygroups of polygroups. In this paper we generalize the notions of left and right commutators in $H_v$-groups. Then using commutators we introduce a new strongly equivalence relation $\pi^*$ on an $H_v$-group $H$ such that the quotient $H/\pi^*$ is a metabelian group. Moreover, we introduce metabelian $H_v$-groups and investigate some of their properties. Finally, we investigate some properties of commutators for the class of weak polygroups. In the following we recall some basic notions of $H_v$-group theory.

Let $H$ be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of $H$. Let $\cdot$ be a hyperoperation (or join operation) on $H$, that is, $\cdot$ is a function from $H \times H$ into $P^*(H)$. If $(a, b) \in H \times H$, its image under $\cdot$ in $P^*(H)$ is denoted by $a \cdot b$. The join operation is extended to subsets of $H$ in a natural way, that is, for non-empty subsets $A, B$ of $H$,

$$A \cdot B = \bigcup \{a \cdot b \mid a \in A, b \in B\}.$$  

The notation $a \cdot A$ is used for $\{a\} \cdot A$ and $A \cdot a$ for $A \cdot \{a\}$. Generally, the singleton $\{a\}$ is identified with its member $a$. The structure $(H, \cdot)$ is called an $H_v$-group if $a \cdot (b \cdot c) \cap (a \cdot b) \cdot c \neq \emptyset$, for all $a, b, c \in H$, which means that

$$\left( \bigcup_{u \in a \cdot b} u \cdot c \right) \cap \left( \bigcup_{v \in b \cdot c} a \cdot v \right) \neq \emptyset$$

and $a \cdot H = H \cdot a = H$ for all $a \in H$. A non-empty subset $K$ of an $H_v$-group $(H, \cdot)$ is called a $H_v$-subgroup if it is an $H_v$-group. Suppose that $(H, \cdot)$ and $(K, \circ)$ are two $H_v$-group. A function $f : H \rightarrow K$ is called a homomorphism if $f(a \cdot b) \subseteq f(a) \circ f(b)$ for all $a$ and $b$ in $H$. We say that $f$ is a good homomorphism if for all $a$ and $b$ in $H$, $f(a \cdot b) = f(a) \circ f(b)$. If $(H, \cdot)$ is an $H_v$-group and $\rho \subseteq H \times H$ is an equivalence, we set

$$A \rho B \iff a \rho b, \quad \forall a \in A, \forall b \in B,$$
for all pairs \((A, B)\) of non-empty subsets of \(H\).

The relation \(\rho\) is called strongly regular on the left (on the right) if 
\[ x \rho y \Rightarrow a \cdot x \equiv a \cdot y \] 
(\(x \rho y \Rightarrow x \cdot a \equiv y \cdot a\), respectively), for all \((x, y, a) \in H^3\). Moreover, \(\rho\) is called strongly regular if it is strongly regular on the right and on the left.

**Theorem 1.1.** If \((H, \cdot)\) is an \(H_v\)-group and \(\rho\) is a strongly regular relation on \(H\), then the quotient \(H/\rho\) is a group under the operation:

\[ \rho(x) \otimes \rho(y) = \rho(z), \text{ for all } z \in x \cdot y. \]

We denote \(\rho(x)\) by \(\bar{x}\) and instead of \(\bar{x} \otimes \bar{y}\) we write \(\bar{x} \bar{y}\). Let \((H, \cdot)\) be an \(H_v\)-group and \(U\) be the set of all finite products of elements of \(H\). For all \(n > 1\), we define the relation \(\beta_n\) on \(H\), as follows:

\[ a \beta_n b \iff \exists u \in U : \{a, b\} \subseteq u, \]

and \(\beta = \bigcup_{i=1}^{n} \beta_n\), where \(\beta_1 = \{(x, x) \mid x \in H\}\) is the diagonal relation on \(H\). Note that, in general, for an \(H_v\)-group may be \(\beta \neq \beta^*\), where \(\beta^*\) is the transitive closure of \(\beta\). The relation \(\beta^*\) is the smallest equivalence relation on an \(H_v\)-group \(H\), such that the quotient \(H/\beta^*\) is a group.

**Definition 1.2.** Let \(H\) be a weak \(H_v\)-group and \(X\) be a non-empty subset of \(H\). We define \(\text{Ass}(X) = \{(x_1, x_2) \in H^2 \mid (x_{\sigma(1)} \cdot x_{\sigma(2)}) \cdot x_{\sigma(3)} = x_{\sigma(1)} \cdot (x_{\sigma(2)} \cdot x_{\sigma(3)}), \forall x_{\sigma} \in X, \forall \sigma \in S_3\}\). Moreover, if \(\text{Ass}(X) = H \times H\), we say that \(X\) is a full associative subset of \(H\).

**Remark 1.3.** An \(H_v\)-group \(H\) is called a hypergroup if and only if \(\text{Ass}(H) = H^2\).

2. ON THE STRONGLY REGULAR RELATION \(\pi^*\)

In this section, we introduce and analyze a new definition of left and right commutators in \(H_v\)-group. Using commutators we introduce a new strongly equivalence relation \(\pi^*\) on an \(H_v\)-group \(H\) such that the set of all equivalence classes; \(\pi^*(x), x \in H\) is a metabelian group. Moreover, we introduce metabelian \(H_v\)-groups and investigate some of their properties.

**Definition 2.1.** Let \((H, \cdot)\) be an \(H_v\)-group and \((x, y) \in H^2\). We define

1. \([x, y]_r = \{h \in H \mid x \cdot y \cap (y \cdot x) \cdot h \cap y \cdot (x \cdot h) \neq \emptyset\}\);
2. \([x, y]_l = \{h \in H \mid x \cdot y \cap h \cdot (y \cdot x) \cap (h \cdot y) \cdot x \neq \emptyset\}\);
3. \([x, y] = [x, y]_r \cup [x, y]_l\).
From now on we call \([x, y]_r\), \([x, y]_l\) and \([x, y]_r\) right commutator \(x\) and \(y\), left commutator \(x\) and \(y\) and commutator \(x\) and \(y\) in \(H\), respectively. Also we will denote \([H, H]_r\), \([H, H]_l\) and \([H, H]\) the sets of all right commutators, left commutators and commutators in \(H\), respectively.

**Example 2.2.** Suppose that \(H = \{0, 1, 2\}\). Consider the \(H_v\)-group \((H, \cdot)\), where \(\cdot\) is defined on \(H\) as follows:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 1,2 & 0,1 & 0 \\
1 & 0,2 & 1 & 2 \\
2 & 0 & 1,2 & 1,2 \\
\end{array}
\]

We can see that \(\{1\} = [1, 1]_r \neq [1, 1]_l = H\).

**Example 2.3.** Suppose that \(H = \{0, 1, 2, 3\}\). Consider the commutative \(H_v\)-group \((H, \cdot)\), where \(\cdot\) is defined on \(H\) as follows:

\[
\begin{array}{ccc}
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 0,2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 0,2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 0,2 \\
\end{array}
\]

In this case we have \([1, 3]_r = \{0, 2\}\) and \([1, 3]_l = \{0\}\) and so \([1, 3]_r \neq [1, 3]_l\).

Notice that the above example is a commutative \(H_v\)-group while \([x, y]_r \neq [x, y]_l \neq [x, y]\), for some \((x, y) \in H^2\).

**Proposition 2.4.** Let \((H, \cdot)\) be a commutative \(H_v\)-group such that \((x \cdot y) \cdot z \subseteq x \cdot (y \cdot z)\), for all \((x, y, z) \in H^3\). Then \([y, x]_r = [x, y]_r = [x, y]_l = [y, x] = [x, y]_r\), for all \((x, y) \in H^2\).

**Proof.** The proof is strightforward. \(\square\)

**Corollary 2.5.** Let \((H, \cdot)\) be a commutative hypergroup. Then, \([y, x]_r = [x, y]_r = [x, y]_l = [y, x] = [x, y]_r\), for all \((x, y) \in H^2\).

Let \((H, \cdot)\) be an \(H_v\)-group, \(n \in \mathbb{N}\) and \((x_1, ..., x_n) \in H^n\). We mean by \(\mathcal{F}(x_1, ..., x_n)\) the set of all finite possible products of \(x_1, ..., x_n\), respectively. For example \(\mathcal{F}(x_1, x_2) = \{x_1 \cdot x_2\}\), \(\mathcal{F}(x_1, x_2, x_3) = \{x_1 \cdot (x_2 \cdot x_3), (x_1 \cdot x_2) \cdot x_3\}\) and \(\mathcal{F}(x_1, x_2, x_3, x_4) = \{(x_1 \cdot (x_2 \cdot x_3)) \cdot x_4, ((x_1 \cdot x_2) \cdot x_3) \cdot x_4, x_1 \cdot (x_2 \cdot x_3) \cdot x_4, (x_1 \cdot x_2) \cdot (x_3 \cdot x_4)\}\). Thus we have \(\mathcal{U} = \bigcup_{(x_1, ..., x_n) \in H^n} \mathcal{F}(x_1, ..., x_n)\). Moreover, suppose that \(u \in \mathcal{F}(x_1, ..., x_n)\) if and only if \(u_\sigma \in \mathcal{F}(x_{\sigma(1)}, ..., x_{\sigma(n)})\), for every \(\sigma \in S_n\).
**Definition 2.6.** Let $H$ be an $H_v$-group. Suppose that $\pi = \bigcup_{m \geq 1} \pi_m$, where $\pi_1$ is the diagonal relation and for every integer $m > 1$, $\pi_m$ is the relation defined as follows:

$$x \pi_m y \iff \exists u \in \mathcal{F}(x_1, \ldots, x_m), \exists \sigma \in S_m : \sigma(i) = i \text{ if } x_i \notin \cup[H, H] \text{ such that } x \in u \text{ and } y \in u\sigma.$$ 

Hence, by the above results, we obtain $x \pi y$ and so $x \pi z \pi y.$

Met Definition 2.8. The $H_v$-group $H$ is called metabelian if and only if $\text{Met}(H) = \omega_H$, where $\omega_H$ is the kernel of the canonical homomorphism $\phi_H : H \to H/\beta^*$; i.e. $\omega_H = \phi^{-1}(1_{H/\beta^*}).$
Example 2.9. Suppose that $H = \{0, 1, 2, 3\}$. Consider the non-commutative $H_v$-group $(H, \cdot)$, where $\cdot$ is defined on $H$ as follows:

\[
\begin{array}{cccc}
  \cdot & 0 & 1 & 2 & 3 \\
0 & 0 & 1, 2 & 2 & 3 \\
1 & 1 & 1 & H & 3 \\
2 & 2 & 0, 1, 2 & 2 & 3 \\
3 & 3 & 1, 3 & 3 & H \\
\end{array}
\]

In this case we can see that $\cup[H, H] = \omega_H = H$, and so $\text{Met}(H) = \omega_H$ which means that $H$ is a metabelian $H_v$-group.

Remark 2.10. If $G$ is a group then $G$ is a metabelian group if and only if $[[x, y], z] = 1_G$, for every $(x, y, z) \in G^3$.

Theorem 2.11. Let $(H, \cdot)$ be an $H_v$-group. Then,

(i) $H/\pi^*$ is a metabelian group;

(ii) $\pi$ is the smallest equivalent relation such that $H/\pi^*$ is a metabelian group.

Proof. (i). According to Theorem 1.1 $H/\pi^*$ is a group. Now let $(\bar{x}, \bar{y}, \bar{z}) \in (H/\pi^*)^3$. We shall prove that $[[\bar{x}, \bar{y}], \bar{z}] = 1_{H/\pi^*}$. To do this suppose that $\bar{a} = [\bar{x}, \bar{y}]$. Without losing the generality we have $x \cdot y \cap (y \cdot x) \cdot a \neq \emptyset$ and so $a \in [H, H]$. Therefore $a \cdot z \pi^* z \cdot a$ so $\bar{a}z = z\bar{a}$. Consequently $[[\bar{x}, \bar{y}], \bar{z}] = 1_{H/\pi^*}$, which means that $\text{Met}(H/\pi^*) = \{1_{H/\pi^*}\} = \omega_H/\pi^*$.

(ii). Suppose that $\rho$ is a strongly regular relation on $H$ such that $H/\rho$ is a metabelian group. Now let $a \pi b$ so there exists $u \in F(x_1, ..., x_m)$, and there exist $\sigma \in S_m$ such that $\sigma(i) = i$ if $x_i \notin [H, H]$ and $x_i \in [H, H]$ or $y \in u_S$. If $x_i \in [H, H]$ then there exists $(s, t) \in H^2$ such that $s \cdot t \cap (t \cdot s) \cdot x_i \cap (s \cdot x_i) \neq \emptyset$ or $s \cdot t \cap x_i \cdot (t \cdot s) \cap (x_i \cdot t) \cdot s \neq \emptyset$. Therefore $\rho(x_i) = [\rho(s), \rho(t)]$ or $\rho(x_i) = [\rho(s), \rho(t)]^{-1}$ and so $\rho(x_i)$ commutes with all elements of $H/\pi^*$. Hence $\rho(a) = \rho(b)$ and so $\pi \subseteq \rho$. Consequently $\pi^* \subseteq \rho$ holds. 

Proposition 2.12. The $H_v$-group $H$ is metabelian if and only if $\beta^* = \pi^*$.

Proof. Suppose that $H$ is a metabelian $H_v$-group. We need to prove that $H/\beta^*$ is a metabelian group. Let $a \in \cup[H, H]$ and $x \in H$. If $y \in [a, x]$ then $y \in \omega_H$. Also we have $a \cdot x \cap (x \cdot a) \cdot y \cap (a \cdot y) \neq \emptyset$. Consequently $\beta^*(a)\beta^*(x) = \beta^*(x)\beta^*(a)$. Thus we have $\beta^* \supseteq \pi^*$ and so $\beta^* = \pi^*$. Conversely suppose that $\beta^* = \pi^*$ we shall prove that $\text{Met}(H) = \bigcup_{x \in [H, H], y \in H} [x, y] \subset \omega_H$. To do this suppose
that $a \in \cup[H, H]$ and $x \in H$. If $y \in [a, x]$, then $a \cdot x \cap (x \cdot a) \cdot y \cap (a \cdot y) \neq \emptyset$. Because $\pi^*(a) \pi^*(x) = \pi^*(x) \pi^*(a)$ we conclude that $\pi^*(y) = \beta^*(y) = 1_{H/\beta}$. Hence $y \in \omega_H$. Similarly if $y \in [a, x]$, we have a similar result. \hfill $\square$

Let $(H, \cdot)$ be an $H_v$-group. Then, we define $Z_v(H) = \{ h | h \cdot h \cdot x \neq \emptyset, \forall x \in H \}$ and we call it the weak center of $H$.

**Theorem 2.13.** Let $(H, \cdot)$ be an $H_v$-group and $\cup[H, H] \subseteq Z_v(H)$. Then, $H$ is metabelian.

**Proof.** Suppose that $(H, \cdot)$ be an $H_v$-group and $\cup[H, H] \subseteq Z_v(H)$, we need to prove that $\pi \subseteq \beta^*$. To do this suppose that $a \pi b$ so there exists $u \in F(x_1, \ldots, x_m)$, and there exists $\sigma \in S_m$ such that $\sigma(i) = i$ if $x_i \notin [H, H]$ and $x \in u$ and $y \in u_\sigma$. By induction on $m$ we show that $u \cap u_\sigma \neq \emptyset$. Therefore $u \cup u_\sigma \neq \emptyset$ and we have $a \beta b$. Consequently $\beta^* = \pi^*$.

Let $(G, \cdot)$ be a group and $R$ be an equivalence relation on $G$. In $\bar{G} = G/R$ consider the hyperoperation $\circ$ defined by $\bar{x} \circ \bar{y} = \{ \bar{z} | z \in \bar{x} \cdot \bar{y} \}$, where $\bar{x}$ denotes the equivalence class of the element $x$. Then, $(\bar{G}, \circ)$ is an $H_v$-group which is not necessary a hypergroup.

**Proposition 2.14.** Let $(G, \cdot)$ be a group and $R$ be an equivalence relation on $G$ and $(\bar{G}, \circ)$ be the associated $H_v$-group. Then, 

$$\overline{[G, G]} = \{ \bar{z} | z \cap [G, G] \neq \emptyset \} \subseteq \cup[\bar{G}, \bar{G}].$$

**Proof.** Let $\bar{a} \in [\bar{G}, \bar{G}]$. Then, $\bar{a} \cap [\bar{G}, \bar{G}] \neq \emptyset$, thus there exist $g \in \bar{a}$ and $x, y \in G$ such that $g = x^{-1} \cdot y^{-1} \cdot x \cdot y$. We show that 

$$\bar{x} \circ \bar{y} \cap (\bar{y} \circ \bar{x}) \cap \bar{a} \cap \bar{y} \circ (\bar{x} \circ \bar{a}) \neq \emptyset.$$ 

We have $y \cdot x \in \bar{y} \cdot \bar{x}$, then $\bar{y} \cdot \bar{x} \in \bar{y} \circ \bar{x}$ and so we have 

$$x \cdot y = (y \cdot x) \cdot x^{-1} \cdot y^{-1} \cdot x \cdot y = (y \cdot x) \cdot g \in \bar{y} \cdot \bar{x} \cdot \bar{a}.$$ 

Therefore $\bar{x} \circ \bar{y} \in (\bar{y} \circ \bar{x}) \circ \bar{a}$. Also, $x \in \bar{x}, y \in \bar{y}$, then $x \cdot y \in \bar{x} \cdot \bar{y}$, so $\bar{x} \circ \bar{y} \in \bar{x} \circ \bar{y}$. Thus $\bar{x} \circ \bar{y} \in \bar{x} \circ \bar{y} \cap (\bar{y} \circ \bar{x}) \circ \bar{a}$. On the other hand $y^{-1} \cdot x \cdot y = x \cdot x^{-1} \cdot y^{-1} \cdot x \cdot y = x \cdot g \in \bar{x} \cdot \bar{a}$, then $y^{-1} \cdot x \cdot \bar{y} \in \bar{x} \circ \bar{a}$. So $x \cdot y = y^{-1} \cdot x \cdot y \in \bar{y} \cdot \bar{y}^{-1} \cdot x \cdot y \subseteq \bar{y} \cdot (\bar{x} \circ \bar{a})$. Thus $\bar{x} \circ \bar{y} \in \bar{y} \circ (\bar{x} \circ \bar{a})$. Therefore 

$$\bar{x} \circ \bar{y} \in \bar{x} \circ \bar{y} \cap (\bar{y} \circ \bar{x}) \cap \bar{a} \cap \bar{y} \circ (\bar{x} \circ \bar{a}) \neq \emptyset.$$
Then \( \bar{a} \in [x, y]_v \), so \( \bar{a} \in \cup \bar{G}, \bar{G} \).

In the following example we show that in Proposition 2.14 the equality does not necessarily hold.

**Example 2.15.** Consider the group \( G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\} \). Let the partition \( \{\{0, 1, 2\}, \{3, 4, 5\}\} \) of \( \mathbb{Z}_6 \). In this case we have \( \bar{G} = G/R = \{0, 3\} \), because

\[
\bar{0} \circ 3 = \{z| z \in \{0, 1, 2\} \cdot \{3, 4, 5\}\} = \{0, 3\}.
\]

For all \( a \in G \) we have

\[
\bar{0} \circ 3 \cap (\bar{0} \circ 3) \cap \bar{3} \cap (\bar{0} \circ \bar{a}) \neq \emptyset.
\]

Therefore, \( \cup \bar{G}, \bar{G} = G = \{0, 3\} \), but \( \bar{3} \cap [G, G] = 3 \cap \{0\} = \emptyset \).

**Definition 2.16.** Let \( R \) be an equivalence relation on \( G \). \( R \) is called good if and only if \( [G, G] \supseteq \cup \bar{G}, \bar{G} \).

**Remark 2.17.** If \( R \) is a good relation on \( (G, \cdot) \), then by Proposition 2.14 we have \( [G, G] = \cup \bar{G}, \bar{G} \).

**Proposition 2.18.** Let \( R \) be an equivalence relation on \( (G, \cdot) \) such that for every \( x \in G \) there exists \( y \in [G, G] \) that \( (x, y) \in R \). Then, \( R \) is a good relation.

**Proof.** Suppose that \( \bar{b} \in [\bar{G}, \bar{G}] \), then \( \bar{b} = \bar{g}, \) where \( g \in [G, G] \). So \( \bar{b} \cap [G, G] \neq \emptyset \). Therefore \( b \in [G, G] \).

**Theorem 2.19.** Let \( (G, \cdot) \) be a metabelian group and \( R \) be a good relation on \( G \). Then \( \bar{G} \) is a metabelian \( H_v \)-group.

**Proof.** Let \( \bar{a} \in \cup \bar{G}, \bar{G} \) and \( \bar{x} \in \bar{G} \), we prove that \( \bar{a} \circ \bar{x} \cap \bar{x} \circ \bar{a} \neq \emptyset \). To do this end, because \( R \) is a good relation we have \( \bar{a} \in \bar{G}, \bar{G} \). So there exists \( g \in [G, G] \) such that \( \bar{a} = \bar{g} \). Because \( G \) is a metabelian group we have \( g \cdot x = x \cdot g \). Consequently \( g \cdot x \in \bar{g} \cdot \bar{x} = \bar{a} \cdot \bar{x} \) and \( x \cdot g \in \bar{x} \cdot \bar{g} = \bar{x} \cdot \bar{a} \), and so \( \bar{g} \cdot \bar{x} \in \bar{a} \circ \bar{x} \cap \bar{x} \circ \bar{a} \) which means \( \bar{a} \circ \bar{x} \cap \bar{x} \circ \bar{a} \neq \emptyset \). So \( \cup \bar{G}, \bar{G} \subseteq Z_v(G) \).

Using Theorem 2.13 we conclude that \( \bar{G} \) is a metabelian \( H_v \)-group.

3. On Metabelian Weak Polygroups

In this section we study the notion of commutators for the class of weak polygroups introduced in [5].

**Definition 3.1.** ([5]) The \( H_v \)-group \( P \) is called a *weak polygroup* and denoted by \( \langle P, \cdot, e, -1 \rangle \), where \( -1 : P \rightarrow P, x \mapsto x^{-1} \) is a map, if the following conditions hold:
Definition 3.4. Let $\langle A, \cdot, e, -1 \rangle$ be a non-empty subset of a weak polygroup $P$. Then

(1) $P$ has a scalar identity $e$; (i.e., $e \cdot x = x \cdot e = x$, for every $x \in P$);
(2) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

Remark 3.2. A weak polygroup $P$ is called a polygroup if and only if $\text{Ass}(P) = P^2$.

Definition 3.3. A non-empty subset $K$ of a weak polygroup $\langle P, \cdot, e, -1 \rangle$ is a weak subpolygroup of $P$ if

(1) $x, y \in K$ implies $x \cdot y \subseteq K$;
(2) $x \in K$ implies $x^{-1} \in K$.

Definition 3.4. Let $X$ be a non-empty subset of a weak polygroup $\langle P, \cdot, e, -1 \rangle$. Let $\{A_i \mid i \in J\}$ be the family of all weak subpolygroups of $P$ which contain $X$. Then $\cap_{i \in J} A_i$ is called the weak subpolygroup generated by $X$. This weak subpolygroup is denoted by $\langle X \rangle$. If $X = \{x_1, x_2, ..., x_n\}$, then the weak subpolygroup $\langle X \rangle$ is denoted by $\langle x_1, x_2, ..., x_n \rangle$. In a special case $\langle \cup P, P \rangle$, $\langle \cup P, P \rangle$ and $\langle \cup [P, P] \rangle$ are denoted by $P', P', P''$, respectively.

Proposition 3.5. Let $\langle P, \cdot, e, -1 \rangle$ be a weak polygroup. Then, for all $(x, y) \in P^2$ we have

\begin{align*}
(1) & \quad [x, y] \subseteq (x^{-1} \cdot y^{-1}) \cdot (x \cdot y) \cap x^{-1} \cdot (y^{-1} \cdot (x \cdot y)); \\
(2) & \quad [x, y] \subseteq (x \cdot y) \cdot (x^{-1} \cdot y^{-1}) \cap ((x \cdot y) \cdot x^{-1}) \cdot y^{-1}.
\end{align*}

Proposition 3.6. Let $\langle P, \cdot, e, -1 \rangle$ be a weak polygroup. Then, $P$ is weak commutative if and only if $e \in [x, y]_r$ (resp. $e \in [x, y]_l$), for all $(x, y) \in P^2$.

Proof. Let $e \in [x, y]_r$, for all $(x, y) \in P^2$. Then, $e \in (x^{-1} \cdot y^{-1}) \cdot (x \cdot y)$, for all $(x, y) \in P^2$ and so $x \cdot y \cap y \cdot x \neq \emptyset$ for all $(x, y) \in P^2$, that means $P$ is weak commutative. The converse is obvious. \hfill $\square$

Proposition 3.7. Let $\langle P, \cdot, e, -1 \rangle$ be a weak polygroup that $x \cdot (y \cdot z) \subseteq (x \cdot y) \cdot z$, for all $(x, y, z) \in P^3$. Then, for all $(x, y) \in P^2$ we have

\begin{align*}
(1) & \quad [x, y]_r = [x^{-1}, y^{-1}]_r; \\
(2) & \quad P' = P' = P''.
\end{align*}

Proof. 1) Let $(x, y) \in P^2$ and $u \in [x, y]_r$. Then, $x \cdot y \cap (y \cdot u \cap y \cdot (x \cdot u) \neq \emptyset$ so there exists $t \in P$ such that $t \in x \cdot y \cap (y \cdot u \cap y \cdot (x \cdot u)$ thus $t \in x \cdot y \cap u \cap u$ for some $v \in y \cdot x$. Since $P$ is a weak polygroup then we have $v^{-1} \in x^{-1} \cdot y^{-1} \cap u \cap (y^{-1} \cdot x^{-1}) \subseteq x^{-1} \cdot y^{-1} \cap u \cdot (y^{-1} \cdot x^{-1}) \cap (u \cdot y^{-1} \cdot x^{-1})$ therefore $u \in [x, y]_r$, hence $[x, y]_r \subseteq [x^{-1}, y^{-1}]_r$. Similarly we have $[x^{-1}, y^{-1}]_r \subseteq [x, y]_r$.
2) Follows from (1). \hfill $\square$
Proposition 3.8. Let \( \langle P, \cdot, e, ^{-1} \rangle \) be a weak polygroup that \( \cup[P, P] \) is a full associative subset of \( P \). Then, for all \((x, y) \in P^2\) we have
\[
(1) \ [x, y]_r = [x^{-1}, y^{-1}]_l;
(2) \ P' = P'_r = P'_l.
\]
Proof. 1) Let \((x, y) \in P^2\) and \(u \in [x, y]_r\). Then, \(x \cdot y \cap (y \cdot x) \cdot u \neq \emptyset\) so there exists \(t \in P\) such that \(t \in x \cdot y \cap (y \cdot x) \cdot u\) thus \(t \in x \cdot y \cap v \cdot u\) for some \(v \in y \cdot x\). Since \(P\) is a weak polygroup then we have \(v^{-1} \in x^{-1} \cdot y^{-1} \cap u \cdot (y^{-1} \cdot x^{-1})\). Therefore \(u \in [x, y]_r\) hence \([x, y]_r \subseteq [x^{-1}, y^{-1}]_l\).
Similarly we have \([x^{-1}, y^{-1}]_l \subseteq [x, y]_r\).
2) Follows from (1). \(\square\)

Corollary 3.9. If \(P\) is a polygroup, then for all \((x, y) \in P^2\) we have
\[
(1) \ [x, y]_r = [x^{-1}, y^{-1}]_l;
(2) \ P' = P'_r = P'_l.
\]

4. CONCLUSION

In this paper we introduce and analyze a generalization of the notion of commutators in \(H_v\)-groups. Several properties are investigated such as introducing a new strongly equivalence relation \(\pi^*\) on an \(H_v\)-group \(H\) such that the quotient \(H/\pi^*\), the set of all equivalence classes, is a metabelian group. This research can be continued, for instance in study of some particular classes of \(H_v\)-groups.

REFERENCES


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